Large-Amplitude Vibrations of Geometrically Imperfect Shallow Spherical Shells with Structural Damping

David Hui*
The Ohio State University, Columbus, Ohio

This paper deals with the effects of geometric imperfections on the large-amplitude vibrations of shallow spherical shells. The initial geometric imperfection, the vibration mode, and the forcing function are of the same spatial shape. It is found that the presence of geometric imperfections of the order of a fractionof the shell thickness may significantly raises the free line vibration frequencies. Further, they may even cause the nonlinear hard-spring character of spherical shells to exhibit soft-spring behavior. The effects of structural damping are also considered.

Introduction

LINEAR and nonlinear (large-amplitude) vibration of shells is a subject that has been extensively investigated in the past decades.[1] Although large-amplitude vibrations of flat rectangular plates[2,3] and cylindrical shells[1] have been examined by many authors, relatively little attention has been devoted to large-amplitude vibrations of spherical shells. Grossman et al.[4] and Leissa and Kadi[5] found that the nonlinearity of shallow spherical shells is of the softening type, at least for amplitudes of vibration up to several times the shell thickness. The nonlinear axisymmetric vibrations of shallow spherical shells was examined by Connor[6] and the dynamic response of spherical shells by Evensen and Fulton.[7]

With the exception of an earlier paper by Hui and Leissa,[8] the effects of geometric imperfections on the free linear and nonlinear vibrations of shallow spherical shells have not been investigated. It has been demonstrated that geometric imperfections of the order of a fraction of the thickness may significantly raise the free linear vibration frequencies of rectangular[9] and circular[10] plates. Furthermore, imperfections of this magnitude may also change the hard-spring behavior to soft-spring behavior. Thus, it can be conjectured that geometric imperfections may also exhibit equally significant effects on the vibrations of shallow spherical shells.

The effects of structural (hysteresis) damping on the vibrations of structures have been studied by a number of authors. The complex-modulus model for structural damping has been discussed in a review article by Bert.[11] It has been applied to the linear forced vibrations of cylindrical shells by Leissa and Iyer[12] and to simply supported rectangular plates[13] and cylindrical panels.[14] In passing, the nonlinear vibrations of simply supported cylindrical shells with no damping was analyzed by Watawala and Nash.[15]

In this paper, the large-amplitude vibration of geometrically imperfect shallow spherical shells with structural damping is examined. The vibration mode, the geometric imperfection, and the forcing function are assumed to have the same spatial shape. Although the shapes of the geometric imperfections are random in practical structures, the introduction of imperfections of specified shape will considerably simplify the theoretical analysis and it will provide useful information on the possible effects in a preliminary design. Following Hutchinson[16] and Reissner[17] the spherical shell is assumed to be shallow and the wavelengths of the vibration modes are assumed to be small compared to the shell radius. The governing differential equations consist of Donnell-type nonlinear dynamic equilibrium and compatibility equations. The nonlinear compatibility equation is satisfied exactly by a suitable choice of the vibration mode and the stress function. The nonlinear dynamic equilibrium equation is then satisfied approximately using the Galerkin procedure. It follows that the nonlinear vibration response is governed by a single second-order nonlinear ordinary differential equation in time. It can be written in terms of Duffing's equation with an additional quadratic term. This equation is solved using the Linstedt's perturbation technique (see Reissner,[18] Cummings,[19] and the Appendix of Ref. 10).

It is found that geometric imperfections of the order of a fraction of the shell thickness may significantly raise the linear free vibration frequencies. Furthermore, they may also cause the usual nonlinear hard-spring character of the spherical shells to exhibit soft-spring behavior. Appropriate backbone curves (graphs of the amplitude of vibration vs the nonlinear vibration frequency) are plotted for various wave numbers. For a given amplitude of vibration, the geometric imperfection may actually lower the nonlinear frequencies. Structural damping is shown to reduce significantly the vibration amplitude near resonance.

Large-Amplitude Vibrations of Imperfect Spherical Shells

The Donnell-type nonlinear dynamic equilibrium and compatibility equation for a shallow, geometrically imperfect spherical shell written in terms of the out-of-plane displacement W and the Airy stress function F, due to Hutchinson[16] and Reissner[17] (see Fig. 1 for the shell geometry), are

$$
\begin{aligned}
&[(1+i\eta)Eh^3/(4c^2)](W,_{XXXX}+W,_{YYYY}+2W,_{XXYY}) \\
&+(1/R)(F,_{XX}+F,_{YY})=Q(X,Y,\bar{t})-\rho W,_{\bar{t}\bar{t}} \\
&+F,_{XX}(W+W_0),_{YY}+F,_{YY}(W+W_0),_{XX} \\
&-2F,_{XY}(W+W_0),_{XY} \qquad (1)
\end{aligned}
$$

$$
\begin{aligned}
&\{1/[Eh(1+i\eta)]\}(F,_{XXXX}+F,_{YYYY}+2F,_{XXYY}) \\
&=(1/R)(W,_{XX}+W,_{YY})+(W,_{XY})^2+2W_0,_{XY}W,_{XY} \\
&-(W+W_0),_{XX}W,_{YY}-W_0,_{YY}W,_{XX} \qquad (2)
\end{aligned}
$$

Received Oct. 29, 1982; revision submitted Jan. 16, 1983.

*Assistant Professor, Department of Engineering Mechanics. Member AIAA.

In the above, W_0 is the initial geometric imperfection (due to, for example, unavoidable manufacturing difficulties), h the shell thickness, R the shell radius, ρ the mass per unit area, $\bar{t}$ the time, $Q(X,Y,\bar{t})$ the forcing function, X and Y the in-plane coordinates, i the complex number $(-1)^{1/2}$, E Young's modulus, η the loss factor associated with the complex-modulus model for structural damping, $c=[3(1-\nu)^2]^{1/2}$, and ν Poisson's ratio. Further, a comma followed by a variable represents partial differentiation with respect to that variable.

It is convenient to define the following nondimensional quantities:

$$(w,w_0)=(W,W_0)/h, \qquad f=2cF/(Eh^3)$$

$$(x,y)=(X,Y)(q_0/R), \qquad q_0=(2cR/h)^{1/2}$$

$$t=\omega_r\bar{t}, \qquad q(x,y,t)=[R^2/(Eh^2)]Q(X,Y,\bar{t}) \tag{3}$$

where ω_r is the reference frequency related to the breathing mode of a perfect, stress-free complete shallow spherical shell,

$$\omega_r=[Eh/(\rho R^2)]^{1/2} \tag{4}$$

Thus, the nonlinear dynamic equilibrium and compatibility equations can be written in nondimensional form,

$$(1+i\eta)(w,_{xxxx}+w,_{yyyy}+2w,_{xxyy})+(f,_{xx}+f,_{yy})$$
$$=q(x,y,t)-w,_{tt}+(2c)[f,_{xx}(w+w_0),_{yy}$$
$$+f,_{yy}(w+w_0),_{xx}-2f,_{xy}(w+w_0),_{xy}] \tag{5}$$

$$[1/(1+i\eta)(f,_{xxxx}+f,_{yyyy}+2f,_{xxyy})=(w,_{xx}+w,_{yy})$$
$$+(2c)\{(w,_{xy})^2+2w_0,_{xy}w,_{xy}-(w+w_0),_{xx}w,_{yy}-w_0,_{yy}w,_{xx}\} \tag{6}$$

The forcing pressure distribution is taken to be of the same shape as the initial geometric imperfection,

$$q(x,y,t)=q(t)\cos(Mx)\cos(Ny) \tag{7a}$$

$$w_0(x,y)=\mu\cos(Mx)\cos(Ny) \tag{7b}$$

where μ is the amplitude of the geometric imperfection normalized with respect to the shell thickness and the wave numbers M and N are defined to be

$$M=m/q_0, \qquad N=n/q_0 \tag{8}$$

where m and n are the number of full sine waves around the circumference of the spherical shell. Following the assumptions employed by Hutchinson,[16] the spherical shell is assumed to be sufficiently shallow in that the wavelength of the vibration mode is small compared with the shell radius. Thus, m and n are no longer required to be integers and the boundary conditions may be replaced by the requirement that the vibration mode be periodic. Thus, the vibration mode will also be sinusoidal of the form,

$$w(x,y,t)=w(t)\cos(Mx)\cos(Ny) \tag{9}$$

where it should be noted that $w(t)$ is in general a complex quantity since the coefficients of the differential equations involve the complex modulus. Substituting the geometric imperfection $w_0(x,y)$ and the vibration mode $w(x,y,t)$ into the compatibility equation, one obtains

$$[1/(1+i\eta)](f,_{xxxx}+f,_{yyyy}+2f,_{xxyy})$$
$$=-(M^2+N^2)w(t)\cos(Mx)\cos(Ny)$$
$$-(cM^2N^2)[w(t)^2+2\mu w(t)][\cos(2Mx)+\cos(2Ny)] \tag{10}$$

Thus, the stress function satisfying the nonlinear compatibility equation exactly is

$$f(x,y,t)=(1+i\eta)\{c_0w(t)\cos(Mx)\cos(Ny)$$
$$+[w(t)^2+2\mu w(t)][c_1\cos(2Mx)+c_2\cos(2Ny)]\} \tag{11}$$

where,

$$c_0=-1/(M^2+N^2)$$
$$c_1=-cN^2/(16M^2)$$
$$c_2=-cM^2/(16N^2) \tag{12}$$

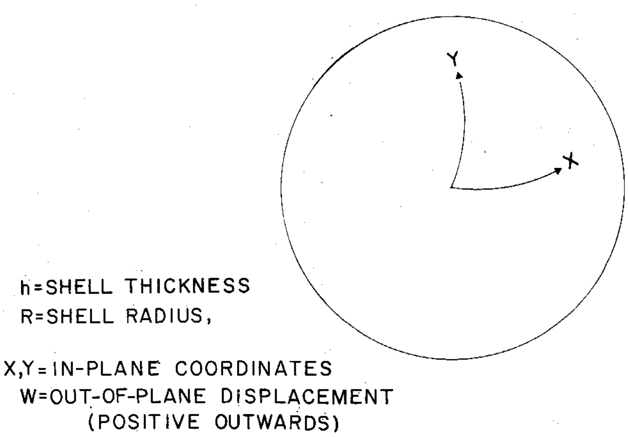

Fig. 1 Geometry of the shallow spherical shell.

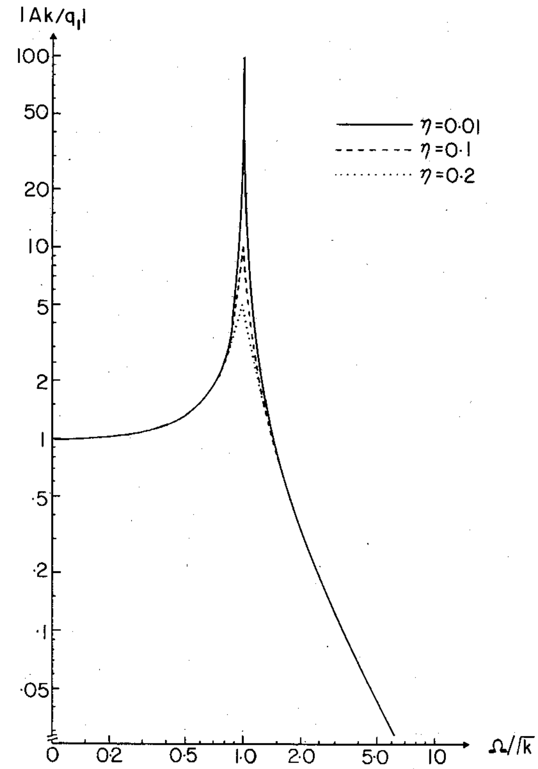

Fig. 2 Normalized linear vibration amplitude ($|Ak/q_1|$) vs normalized forcing frequency ($\Omega/\sqrt{k}$) for various values of the loss factor η.

Finally, substituting $w(x,y,t)$, $w_0(x,y)$, and $f(x,y,t)$ into the nonlinear dynamic equilibrium equation and applying the Galerkin procedure, that is, multiplying both sides by $\cos(Mx)$ $\cos(Ny)$, integrating over the spherical shell surface, and employing the simplifying assumption that the characteristic wavelength of the vibration mode is small compared with the radius, one obtains

$$w(t)_{,tt} + (1+i\eta)\,[kw(t) + (\epsilon k a_2)\,w(t)^2 + (\epsilon k)\,w(t)^3] = q(t) \tag{13}$$

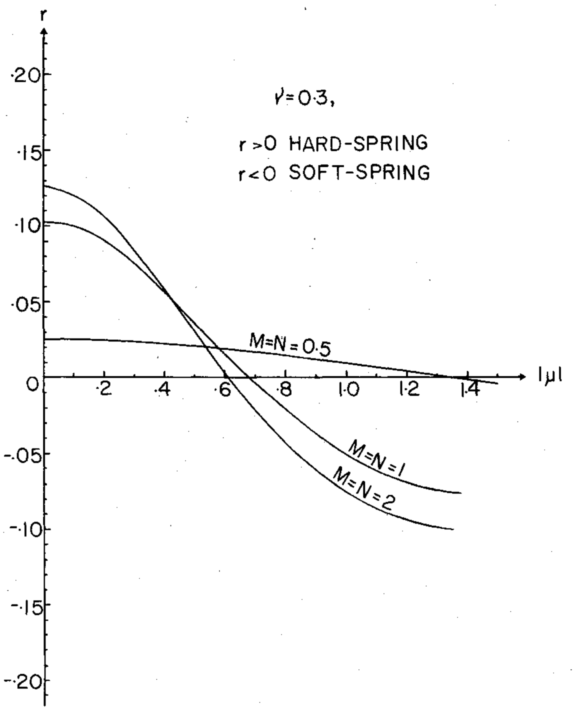

Fig. 3a Nonlinearity parameter vs imperfection amplitude for nonlinear vibration of spherical shells ($\nu=0.3$; $M=N=0.5$, 1, and 2).

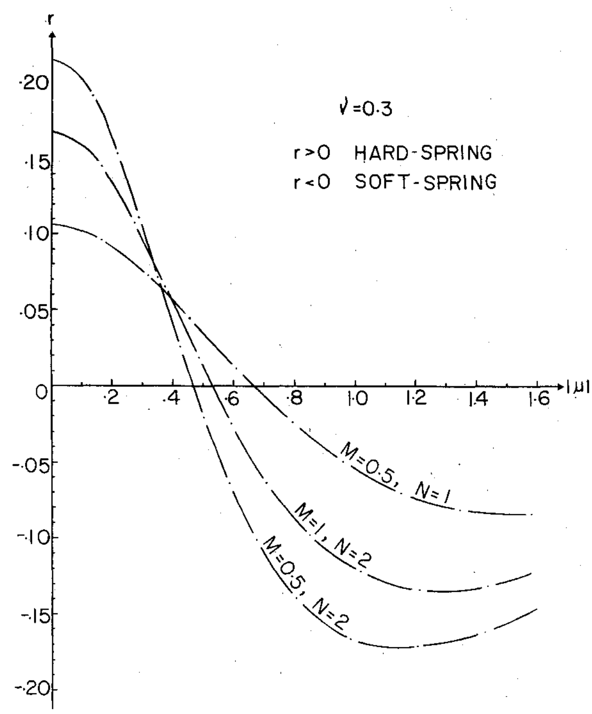

Fig. 3b Nonlinearity parameter vs imperfection amplitude for nonlinear vibration of spherical shells [$\nu=0.3$; ($M=0.5$, $N=1$), ($M=1$, $N=2$), and ($M=0.5$, $N=2$)].

where k, ϵ, and a_2 are defined to be

$$k = 1 + (M^2+N^2)^2 + (M^4+N^4)\,(\mu^2c^2/2) \tag{14a}$$

$$\epsilon = (M^4+N^4)\,[c^2/(4k)] \tag{14b}$$

$$a_2 = 3\mu \tag{14c}$$

The above differential equation is written in terms of the well-known Duffing's equation with an additional quadratic term. It should be noted that the linear free vibration frequency is,

$$\Omega_0 = k^{1/2} \tag{15}$$

and the value for Ω_0 for the special case of a perfect shallow spherical shell agrees with the earlier analysis.[8] The solution of the linearized (that is, neglecting quadratic and cubic terms) differential equation is

$$q(t) = q_1\cos(\Omega t), \qquad w(t) = A\cos(\Omega t) \tag{16}$$

where $\Omega t = \omega\bar{t} = \omega t/\omega_r$, and the absolute value of the complex variable A is

$$|A| = \frac{(q_1/k)}{\{[1-(\Omega^2/k)]^2+\eta^2\}^{1/2}} \tag{17}$$

The backbone curves for large-amplitude vibrations of shallow spherical shells can be computed by solving the nonlinear second-order ordinary differential equation with no damping and no forcing term. Using Linstedt's perturbation method (see Appendix A of Ref. 10), the ratio of the nonlinear to the linear frequency is related to the vibration amplitude A by

$$\Omega/\Omega_0 = 1 + rA^2 - (15\epsilon^2A^4/256) \tag{18}$$

where

$$r = (3\epsilon/8) - (5a_2^2\epsilon^2/12) = (3\epsilon/8)\,[1-(10\mu^2\epsilon)] \tag{19}$$

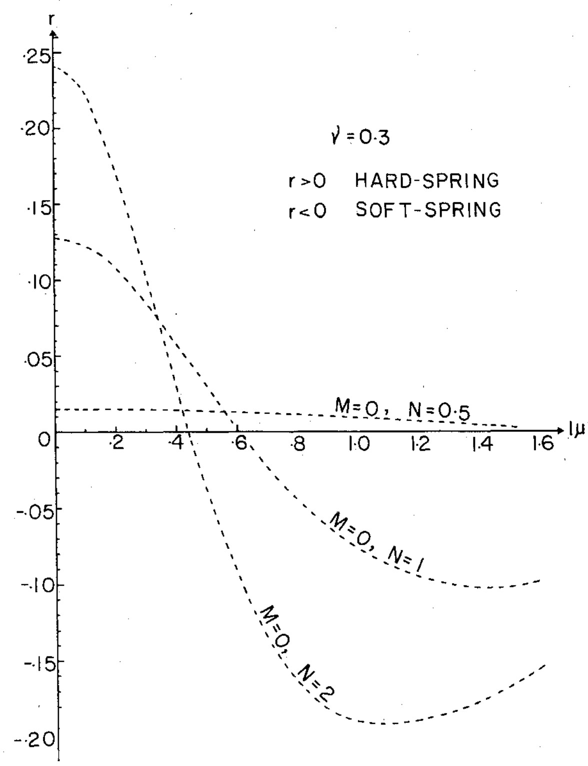

Fig. 3c Nonlinearity parameter vs imperfection amplitude for nonlinear vibration of spherical shells [$\nu=0.3$; ($M=0$, $N=0.5$), ($M=0$, $N=1$), and ($M=0$, $N=2$)].

Thus, the nonlinear vibration of spherical shells can be classified as hard- or soft-spring, depending on whether the nonlinearity parameter r is, respectively, positive or negative (at least for small values of the amplitude $|A|$). Setting r to zero, the transition imperfection amplitude is

$$\mu_t = \left(\frac{1 + (M^2+N^2)^2}{(2c^2)(M^4+N^4)} \right)^{1/2} \tag{20}$$

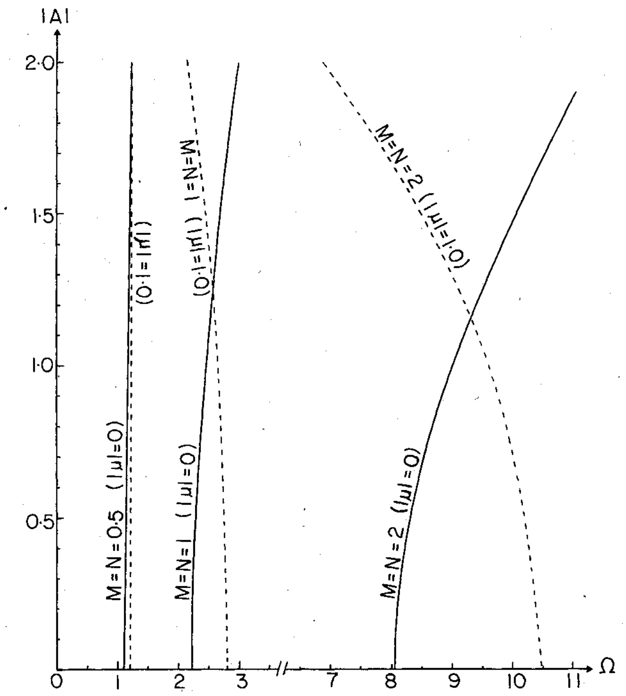

Fig. 4a Backbone curves for nonlinear free vibration of spherical shells ($\nu=0.3$; $M=N=0.5$, 1, and 2; $|\mu|=0$ and 1.0).

Thus, for the magnitude of the geometric imperfection greater than μ_t, the nonlinear vibration problem is of the soft-spring type.

Discussion of Results

Figure 2 is a graph of the vibration amplitude normalized with respect to its static value ($|Ak/q_l|$) vs the vibration frequency normalized with respect to the linear free vibration frequency ($\Omega/\sqrt{k}$) for practical values of the loss factor η associated with structural damping values of 0.01, 0.1, and 0.2. It can be seen that the peaks of these curves remain unshifted with respect to frequency and that the peak value is $1/\eta$.

Figure 3a plots the nonlinearity parameter r vs the imperfection amplitude for nonlinear free vibrations of shallow spherical shells with Poisson's ratio $\nu=0.3$. The wave numbers M and N are equal and take the values $M=N=0.5$, 1, and 2. A similar plot for wave numbers ($M=0.5$, $N=1$), ($M=1$, $N=2$), and ($M=0.5$, $N=2$) are presented in Fig. 3b. For the case of unidirectional vibration, a graph of the nonlinearity parameter vs the imperfection amplitude is shown in Fig. 3c for wave numbers ($M=0$, $N=0.5$), ($M=0$, $N=1$), and ($M=0$, $N=2$). It can be seen that the nonlinearity parameter decreases and then slightly increases with increasing values of the imperfection amplitude. The nonlinear vibration is classified (at least for small values of $|A|$) as hard- or soft-spring depending on whether r is, respectively, greater than or less than zero. Due to the symmetry of the problem, interchanging the values of wave numbers M and N will not affect the results. Furthermore, a sign change in the imperfection amplitude has no effect on the analysis. Analogous to an earlier buckling paper by Hutchinson[16] the present results are independent of the radius-to-thickness ratio, provided wave numbers M and N are specified. Note that the wave numbers of the vibration mode are taken to be the same as the imperfection wave numbers, a situation quite different from the analysis presented in Ref. 8.

Figure 4a shows the backbone curves for nonlinear free vibrations of shallow spherical shells with no damping. Again, Poisson's ratio is 0.3 and the wave numbers under consideration are $M=N=0.5$, 1, and 2. It can be seen that the

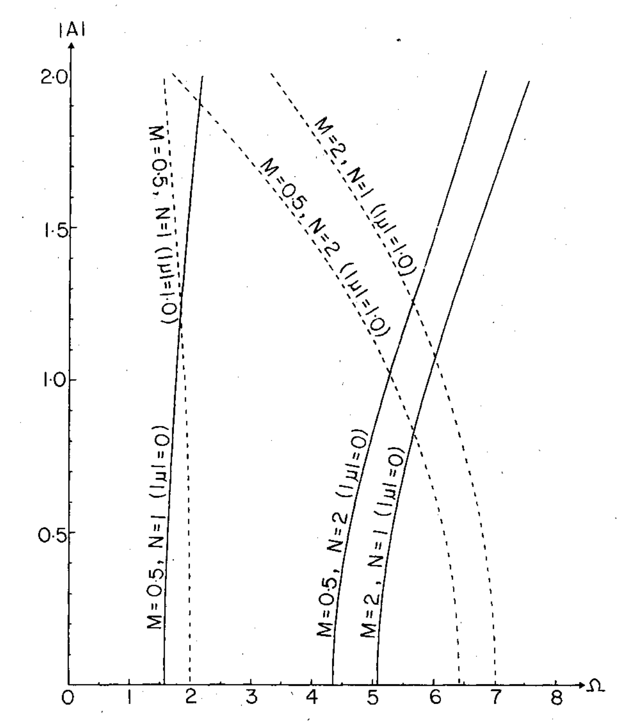

Fig. 4b Backbone curves for nonlinear free vibration of spherical shells [$\nu=0.3$; ($M=0.5$, $N=1$), ($M=0.5$, $N=2$), and ($M=2$, $N=1$); $|\mu|=0$ and 1.0].

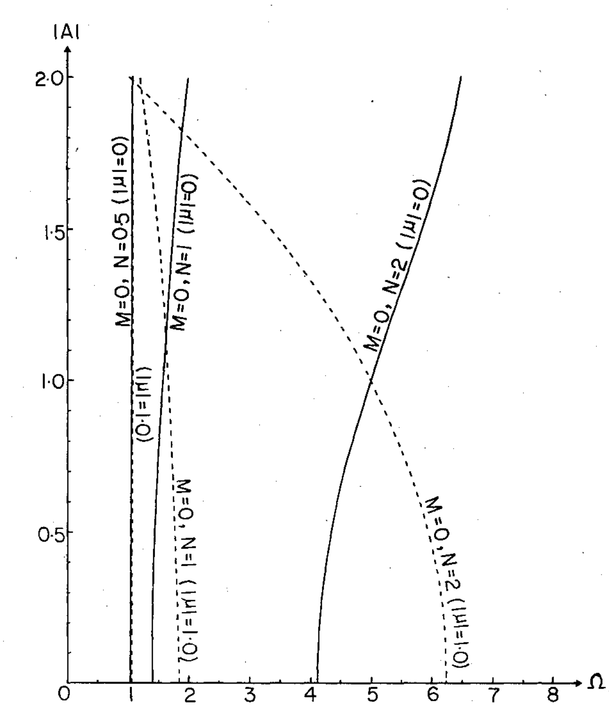

Fig. 4c Backbone curves for nonlinear free vibration of spherical shells [$\nu=0.3$; $M=0$, $N=0.5$). ($M=0$, $N=1$), and ($M=0$, $N=2$); $|\mu|=0$ and 1.0].

presence of geometric imperfection may significantly raise the free linear vibration frequency (corresponding to $|A|=0$). The effect is less pronounced when both M and N are small. Furthermore, the nonlinear vibrations of perfect ($\mu=0$) spherical shells is of the hard-spring type and the corresponding "positive" values of the nonlinearity parameter r can be found from Fig. 3a. On the other hand,

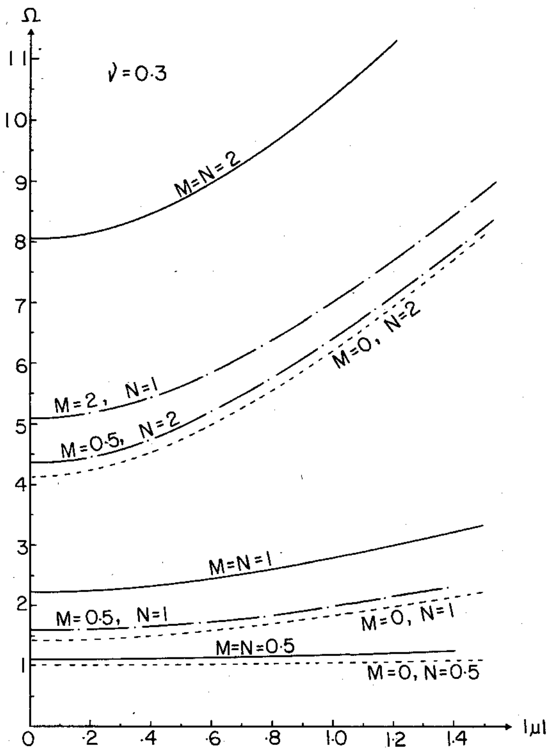

Fig. 5a Linear free vibration frequency vs imperfection amplitude for spherical shells with various wave numbers ($\nu=0.3$).

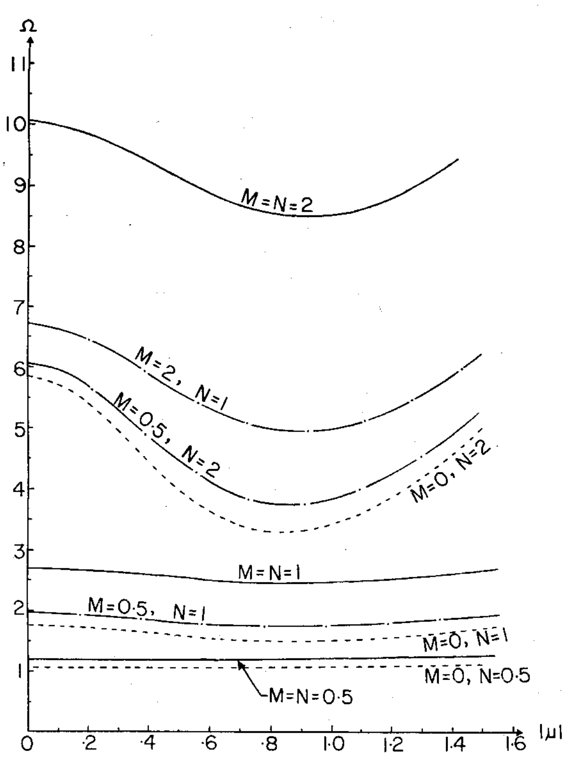

Fig. 5b Nonlinear free vibration frequency vs imperfection amplitude for spherical shells ($\nu=0.3$ and $|A|=1.5$).

nonlinear vibrations of imperfect ($\mu=1.0$) spherical shells with wave numbers ($M=N=2$) and ($M=N=1$) are of the soft-spring type. Again, the corresponding "negative" values of r can be found in Fig. 3a. The backbone curve for the wave numbers $M=N=0.5$ remains more or less vertical for $\mu=1.0$ (slightly curved to the right since r is a small but positive quantity).

Similarly, the backbone curves for the nonlinear free vibrations of shallow spherical shells with wave numbers ($M=0.5$, $N=1$), ($M=1$, $N=2$), and ($M=0.5$, $N=2$) are shown in Fig. 4b. Those curves associated with unidirectional vibrations, that is, with wave numbers ($M=0$, $N=0.5$), ($M=0$, $N=1$), and ($M=0$, $N=2$), are shown in Fig. 4c. Again, Poisson's ratio is 0.3 and interchanging the values for wave numbers M and N will not affect the problem. The nonlinear vibrations of perfect spherical shells are all of the hard-spring type and, except for the case where M and N are small ($M=0$, $N=0.5$), the nonlinear vibrations of imperfect spherical shells with amplitude $\mu=1.0$ are of the soft-spring type.

Finally, a plot of the linear free vibration frequency (corresponding to $|A|=0$) vs the imperfection amplitude for shallow spherical shells with a Poisson's ratio of 0.3 for various values of M and N is presented in Fig. 5a. As expected from Eq. (14a), the effect of a geometric imperfection [of the shape specified by Eq. (7b)] is to raise the free linear vibration frequency. This effect is less pronounced if both M and N are small. On the other hand, for a fixed value of the vibration amplitude $|A|=1.5$, the frequency may actually decrease with increasing imperfection amplitude due to the nonlinear soft-spring character of imperfect spherical shells (see Fig. 5b). The eventual rise of these curves is due to two factors: the nonlinearity parameter r becomes less negative and the free linear vibration frequency continues to increase with increasing imperfection amplitude. Again, it should be noted that the frequency is *not* normalized with respect to its linear ($|A|=0$) value; rather, it is normalized with respect to ω_r.

A direct comparison with the results for nonlinear vibrations of shallow spherical shells with clamped circular boundaries[4] and simply supported boundaries with rectangular planforms[5] is not possible due to the assumption in the present paper that the wavelengths of the vibration modes are small compared to the shell radius. The present paper is applicable to a shallow section of a spherical shell[16] as well as a complete sphere.

Conclusions

The effects of geometric imperfections on the large-amplitude vibrations of shallow spherical shells have been examined. It was found that geometric imperfections of the order of a fraction of the shell thickness may raise significantly the linear free vibration frequencies. Furthermore, they may cause the spherical shells to display soft-spring behavior instead of the inherent hard-spring behavior. Structural damping is found to reduce significantly the vibration amplitudes near resonance.

Finally, it should be mentioned that the present method of analysis can also be applied to a geometric imperfection of an arbitrarily specified shape. The "general" imperfection can be expanded in terms of a Fourier series in the x and y directions. The subsequent analysis will be complicated by the fact that the vibration mode will generally involve more terms, so that the Galerkin procedure will involve minimization with respect to appropriate terms in the vibration mode. The analysis that involves the effects of "general" imperfections is beyond the scope of the present investigation.

Acknowledgment

This research was initiated by the author while he was a graduate student at the Department of Ocean Engineering, Massachusetts Institute of Technology.

References

[1]Leissa, A. W., "Vibration of Shells," NASA SP-288, 1973, 434 pp.

[2]Prathap, G. and Varadan, T. K., "On the Non-Linear Vibrations of Rectangular Plates," *Journal of Sound and Vibration*, Vol. 56, No. 4, pp. 521-530; also Letter to the Editor, *Journal of Sound and Vibration*, Vol. 61, No. 2, pp. 311-314.

[3]Chia, C. Y., *Nonlinear Analysis of Plates*, McGraw-Hill Book Co., New York, 1980, 432pp.

[4]Grossman, P. L., Koplik, B., and Yu, Y. Y., "Nonlinear Vibrations of Shallow Spherical Shells," *Journal of Applied Mechanics*, Sept. 1969, pp. 451-458.

[5]Leissa, A. W. and Kadi, A. S., "Curvature Effects of Shallow Shell Vibrations," *Journal of Sound and Vibration*, Vol. 16, No. 2, 1971, pp. 173-187.

[6]Connor, J. Jr., "Nonlinear Transverse Axisymmetric Vibrations of Shallow Spherical Shells," NASA TN-D1510, 1962, pp. 623-642.

[7]Evensen, D. A. and Fulton, R. E., "Some Studies on the Nonlinear Dynamic Response of Shell-Type Structures," NASA TMX-56843, Oct. 1965; *Dynamic Stability of Structures*, Edited by G. Herrmann, Pergamon Press, New York pp. 237-254.

[8]Hui, D. and Leissa, A. W., "Effects of Uni-Directional Imperfections on Vibrations of Pressurized Shallow Spherical Shells," submitted to *International Journal of Nonlinear Mechanics*, 1983, in press; *Proceedings of the 19th Meeting of the Society of Engineering Sciences*, University of Missouri at Rolla, Oct. 27-29, 1982, p. 314.

[9]Hui, D. and Leissa, A. W., "Effects of Geometric Imperfections on Vibrations of Biaxially Compressed Rectangular Plates," *Journal of Applied Mechanics*, 1983 (ASME Paper, Winter Annual Meeting, Boston, Mass., Nov. 1983).

[10]Hui, D., "Influence of Geometric Imperfections on Large Amplitude Axisymmetric Vibrations of Geometrically Imperfect Circular Plates," *Journal of Sound and Vibration*, Vol. 19, No. 3, 1983 (provisionally scheduled).

[11]Bert, C. W., "Material Damping: An Introductory Review of Mathematical Models, Measures and Experimental Techniques," *Journal of Sound and Vibration*, Vol. 29, No. 2, 1973, pp. 129-153.

[12]Leissa, A. W. and Iyer, K. M., "Modal Response of Circular Cylindrical Shells with Structural Damping," *Journal of Sound and Vibration*, Vol. 77, No. 1, 1981, pp. 1-10.

[13]Hui, D., "Effects of Geometric Imperfections on Large Amplitude Vibrations of Rectangular Plates with Hysteresis Damping," *Journal of Applied Mechanics*, 1984, in press.

[14]Hui, D., "Influence of Geometric Imperfections and In-Plane Constraints on Nonlinear Vibrations of Simply Supported Cylindrical Panels," *Journal of Applied Mechanics*, 1984 (ASME Paper, Summer Applied Mechanics Conference, San Antonio, Texas, June 1984).

[15]Watawala, L. and Nash, W., "Influence on Initial Geometric Imperfections of Vibrations of Thin Circular Cylindrical Shells," *Computers and Structures*, Vol. 16, No. 1-4, 1983, pp. 125-230.

[16]Hutchinson, J. W., "Imperfection Sensitivity of Externally Pressurized Spherical Shells," *Journal of Applied Mechanics*, March 1967, pp. 49-55.

[17]Reissner, E., "A Note on Postbuckling Behavior of Pressurized Shallow Spherical Shells," *Journal of Applied Mechanics*, June 1970, pp. 533-534.

[18]Reissner, E., "Non-Linear Effects in Vibrations of Cylindrical Shells," The Wooldridge Corp., Guided Missile Research Div., Aeromechanics Sec. Rept. AM 5-6, Sept. 1955.

[19]Cummings, B. E., "Some Nonlinear Vibration and Response Problems of Cylindrical Panels and Shells," Aeroelasticity and Structural Dynamics, GALCIT, California Institute of Technology, Rept. SM-62-32 (AFOSR 3123), June 1962.